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On Fourier Series of a Discrete Jacobi–Sobolev Inner Product

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Let μ be the Jacobi measure supported on the interval $[-1, 1]$ and introduce the discrete Sobolev-type inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, d\mu(x) + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k) g^{(i)}(a_k),$$

where $a_k,\ 1 \leqslant k \leqslant K$, are real numbers such that $|a_k| > 1$ and $M_{k,i} > 0$ for all k, i. This paper is a continuation of Marcellan *et al.* (On Fourier series of Jacobi–Sobolev orthogonal polynomials, J. Inequal. Appl., to appear) and our main purpose is to study the behaviour of the Fourier series associated with such a Sobolev inner product. For an appropriate function f, we prove here that the Fourier–Sobolev series converges to f on $(-1, 1) \bigcup_{k=1}^{K} \{a_k\}$, and the derivatives of the series converge to $f^{(i)}(a_k)$ for all i and k. Roughly speaking, the term appropriate means here the same as we need for a function f in order to have convergence for its Fourier series associated with the standard inner product given by the measure μ. No additional conditions are needed.

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1. INTRODUCTION

Let μ be a finite positive Borel measure on the interval $[-1, 1]$ such that *supp* μ is an infinite set and let a_k, for $k = 1, \ldots, K$, be real numbers such that $|a_k| > 1$. For f and g in $L^2(\mu)$ such that there exist the derivatives in a_k, we can introduce the Sobolev-type inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, d\mu(x) + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k) g^{(i)}(a_k), \tag{1}$$

where $M_{k,i} > 0$ for $i = 0, \ldots, N_k$ and $k = 1, \ldots, K$. Let $(\hat{B}_k(x))_{k=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to this inner product,

$$\langle \hat{B}_n, \hat{B}_k \rangle = \delta_{n,k}, \quad k, n = 0, 1, \ldots \ .$$

For every function f such that $\langle f, \hat{B}_k \rangle$ exists for $k = 0, 1, \ldots$, we introduce the formal associated Fourier–Sobolev series

$$\sum_{k=0}^{\infty} \langle f, \hat{B}_k \rangle \hat{B}_k(x).$$

In this paper, we continue the work presented in [3] and its main purpose is to prove the relations

$$\sum_{k=0}^{\infty} \langle f, \hat{B}_k \rangle \hat{B}_k(x) = f(x), \quad x \in (-1, 1),$$

$$\sum_{k=0}^{\infty} \langle f, \hat{B}_k \rangle \hat{B}_k^{(i)}(a_k) = f^{(i)}(a_k), \quad 0 \leqslant i \leqslant N_k, \quad 1 \leqslant k \leqslant K,$$

under standard sufficient conditions for f when the Jacobi measure, $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}\, dx$, $\alpha > -1$, $\beta > -1$, is considered. The precise terms of this result are given in Section 4.

In order to obtain it, we previously need some estimates for the polynomials $\hat{B}_n(x)$ in $[-1, 1] \bigcup_{k=1}^{K} \{a_k\}$ as well as for the involved derivatives $\hat{B}_n^{(i)}(a_k)$. These estimates are studied in Section 3 not only for the Jacobi measure but also for every measure μ belonging to the Szegö class. We start with a representation of $\hat{B}_n(x)$ in terms of the polynomials $(q_n(x))_{n=0}^{\infty}$ which are orthonormal with respect to the measure $w_N(x)\, d\mu(x)$, where $w_N(x)$ is a polynomial with zeros of multiplicity $N_k + 1$ at the points a_k and $N = \sum_{k=1}^{K} (N_k + 1)$ is the degree of $w_N(x)$. In Section 2, we prove that

$$\hat{B}_n(x) = \sum_{i=0}^{N} A_{n,i} q_{n-i}(x)$$

and the sequences $(A_{n,i})_{n=0}^{\infty}$, $i = 0, \ldots, N$, are convergent when the measure μ is such that $\mu' > 0$ a.e. One consequence of this result is the strong and ratio asymptotics for the polynomials $\hat{B}_n(x)$. The relative asymptotics are well known from papers by López, *et al.* [2] and Marcellán and Van Assche [5], where they solved this problem using a different representation of the polynomials $\hat{B}_n(x)$.

In order to obtain some results, we will use the *auxiliary* space

$$\mathbb{S} = \{f : f \in L^2(\mu),\ \ f^{(i)}(a_k) \text{ exists},\ \ i = 0, \ldots, N_k,\ \ k = 1, \ldots, K\}$$

with the inner product (1), where f is assumed to be defined in a neighbourhood of a_k and its derivatives are considered in the ordinary sense. The space $\mathbb{S}$ behaves like a vector space with one component in $L^2(\mu)$ and a finite number of real components.

The fact that the points a_k are outside the interval $[-1, 1]$ plays an important role in the whole paper because, in this case, $\frac{1}{w_N(x)}$ is a continuous function in that interval. Note that some estimates of the polynomials $\hat{B}_n$, when a mass point at $a = 1$ is considered, have been obtained in [1]. The problem of the estimates and the behaviour of the Fourier series when the mass points a_k lie on the interval $[-1, 1]$ remains open.

2. AUXILIARY RESULTS

Let $N = \sum_{k=1}^{K}(N_k + 1)$ and let $w_N(x)$ be the polynomial

$$w_N(x) = \prod_{k=1}^{K} (x - a_k)^{N_k+1}.$$

In order to have positivity for $w_N(x)$ and also to make the notation more comfortable, without loss of generality, we will assume that all points a_k belong to the interval $(-\infty, -1)$; otherwise, we only have to change the corresponding factor $(x - a_k)$ by $(a_k - x)$ in the definition of $w_N(x)$.

Let us consider the polynomials

$$\{1, x - a_1, (x - a_1)^2, \ldots, (x - a_1)^{N_1+1}, (x - a_1)^{N_1+1}(x - a_2), \ldots, (x - a_1)^{N_1+1}$$

$$(x - a_2)^{N_2+1}, \ldots, (x - a_1)^{N_1+1}(x - a_2)^{N_2+1}, \ldots, (x - a_K)^{N_K}\}$$

and denote them as $w_{k,1}(x)$ for $k = 0, \ldots, N - 1$. It is clear that they constitute a basis of the vector space $\mathbb{P}_{N-1}$ of the polynomials of degree less than N. Let $w_{N-k,2}(x)$ be such that $w_{k,1}(x)w_{N-k,2}(x) = w_N(x)$.

Let $(q_n(x))_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to the measure $w_N(x)\,d\mu(x)$ and let

$$\Pi_N(x) = 1 + \sum_{k=1}^{N} b_k T_k(x),$$

where $T_k(x) = \cos k\theta,\ x = \cos\theta$ are the Tchebichef polynomials of the first kind, the Nth polynomial orthogonal with respect to $\frac{1}{\pi}\frac{dx}{w_N(x)\sqrt{1-x^2}}$. We will also denote $\kappa(\pi)$ the leading coefficient of any polynomial $\pi(x)$.

LEMMA 2.1. *For $n \geqslant N$, there exist constants $A_{n,i}$ such that*

$$\hat{B}_n(x) = \sum_{i=0}^{N} A_{n,i} q_{n-i}(x), \quad A_{n,N} \neq 0.$$

If the measure μ is such that $\mu'(x) > 0$ a.e. in $[-1, 1]$, *then* $\lim_{n\to\infty} A_{n,i} = A_i$, *where*

$$A_0 = \frac{1}{\sqrt{2^N b_N}}, \qquad A_i = \frac{b_i}{\sqrt{2^N b_N}}, \quad 1 \leqslant i \leqslant N.$$

Proof. Since $\hat{B}_n(x) = \sum_{j=0}^{n} A_{n,j} q_{n-j}(x)$ and

$$A_{n,j} = \int_{-1}^{1} \hat{B}_n(x) q_{n-j}(x) w_N(x)\, d\mu(x) = \langle \hat{B}_n, q_{n-j} w_N \rangle = 0, \quad N < j \leqslant n,$$

taking into account that $A_{n,N} = \langle \hat{B}_n, q_{n-N} w_N \rangle \neq 0$, the first assertion holds.

On the other hand, from the orthonormality of $\hat{B}_n(x)$, we get

$$\sum_{i=0}^{N} A_{n,i}^2 = \int_{-1}^{1} \hat{B}_n^2(x) w_N(x)\, d\mu(x) \leqslant \max_{x\in[-1,1]} |w_N(x)|$$

and, as a consequence, $|A_{n,i}|$ are bounded. Moreover,

$$A_{n,0} = \frac{\kappa(\hat{B}_n)}{\kappa(q_n)}, \qquad A_{n,N} = \langle \hat{B}_n(x), w_N(x) q_{n-N}(x) \rangle = \frac{\kappa(q_{n-N})}{\kappa(\hat{B}_n)} = \frac{\kappa(q_{n-N})}{\kappa(q_n)} \frac{1}{A_{n,0}}. \tag{2}$$

Also from the orthonormality of $\hat{B}_n(x)$, the sequences $(\hat{B}_n^{(i)}(a_k))_{n=0}^{\infty}$ for $0 \leqslant i \leqslant N_k,\ k = 1, \ldots, K$ are bounded.

Let Λ be a family of non-negative integers such that $(A_{n,i})_{n\in\Lambda}$ is convergent for each $i = 0, 1, \ldots, N$ and let $A_i = \lim_{n\in\Lambda} A_{n,i}$. As it is well known (see

[6, 7]), the condition $\mu' > 0$ a.e. gives ratio asymptotics and the equalities

$$\lim_{n\to\infty} \frac{\kappa(q_{n-N})}{\kappa(q_n)} = \frac{1}{2^N},$$

$$\lim_{n\to\infty} \int_{-1}^{1} f(x) q_n(x) q_{n-k}(x) w_N(x)\, d\mu(x) = \frac{1}{\pi} \int_{-1}^{1} f(x) T_k(x) \frac{dx}{\sqrt{1-x^2}}$$

for any continuous function $f(x)$ also hold. As a consequence, taking into account that $\frac{1}{w_N(x)}$ is a continuous function on $[-1, 1]$,

$$\begin{aligned}
\lim_{n\in\Lambda} \int_{-1}^{1} \hat{B}_n(x) q_n(x) w_{k,1}(x)\, d\mu(x) &= \lim_{n\in\Lambda} \sum_{i=0}^{N} A_{n,i} \int_{-1}^{1} q_{n-i}(x) q_n(x) \frac{w_N(x)}{w_{N-k,2}(x)}\, d\mu(x) \\
&= \sum_{i=0}^{N} A_i \frac{1}{\pi} \int_{-1}^{1} \frac{T_i(x)}{w_{N-k,2}(x)} \frac{dx}{\sqrt{1-x^2}} \\
&= \frac{1}{\pi} \int_{-1}^{1} \sum_{i=0}^{N} A_i T_i(x) w_{k,1}(x) \frac{dx}{w_N(x)\sqrt{1-x^2}}.
\end{aligned}$$

Since $A_N = \frac{1}{2^N A_0}$, if we prove that $\lim_{n\to\infty} \int_{-1}^{1} \hat{B}_n(x) q_n(x) w_{k,1}(x)\, d\mu(x) = 0$ for $k = 0, \ldots, N-1$, the statement of the lemma follows because this means that $\sum_{i=0}^{N} A_i T_i(x)$ is an orthogonal polynomial of degree N with respect to $\frac{1}{\pi} \frac{dx}{w_N(x)\sqrt{1-x^2}}$, and, since $A_0 > 0$ because $A_N < \infty$, $1 + \sum_{i=1}^{N} \frac{A_i}{A_0} T_i(x) = \Pi_N(x)$. Let us prove the previous assertion.

Consider the basis of $\mathbb{P}$

$$\{1, w_{1,2}(x), w_{2,2}(x) \ldots, w_{N-1,2}(x), w_{N,2}(x) = w_N(x), x w_N(x), \ldots\}.$$

If we write $\hat{B}_n(x)$ in terms of this basis, we have

$$\hat{B}_n(x) = \sum_{i=0}^{N} \alpha_{n,i} w_{i,2}(x) + w_N(x) \sum_{i=1}^{n-N} \beta_{n,i} x^i,$$

where $w_{0,2}(x) = 1$. There exists a constant C, independent of n, such that $|\alpha_{n,i}| \leqslant C$ for $i = 0, 1, \ldots, N-1$, because $\alpha_{n,0} = \hat{B}_n(a_K)$ which is bounded as we already know, and if we assume that $\alpha_{n,0}, \alpha_{n,1}, \ldots, \alpha_{n,i}$ are proved bounded, since

$$w_{i+1,2}(x) = (x - \xi) w_{i,2}(x) \qquad \text{for } \xi \in \{a_1, \ldots, a_K\},$$

we have one of the following two possibilities:

First, writing $w_{i+1,2}(x) = (x - \xi)^v \pi(x)$ for a polynomial $\pi(x)$ such that $\pi(\xi) \neq 0$, when v is less than the multiplicity of ξ as a zero of $w_N(x)$,

$$\alpha_{n,i+1} = \lim_{x \to \xi} \frac{\hat{B}_n(x) - \sum_{t=0}^{i} \alpha_{n,t} w_{t,2}(x)}{w_{i+1,2}(x)} = \frac{\hat{B}_n^{(v)}(\xi) - \sum_{t=0}^{i} \alpha_{n,t} w_{t,2}^{(v)}(\xi)}{v!\pi(\xi)}.$$

Second, when v is equal to such a multiplicity, denoting ξ^* the consecutive zero in the construction of the $w_{i,2}(x)$,

$$\alpha_{n,i+1} = \frac{\hat{B}_n(\xi^*) - \sum_{t=0}^{i} \alpha_{n,t} w_{t,2}(\xi^*)}{w_{i+1,2}(\xi^*)}.$$

In both cases $\alpha_{n,i+1}$ is bounded because $(\hat{B}_n^{(v)}(\xi))_{n=0}^{\infty}$ and $(\hat{B}_n(\xi^*))_{n=0}^{\infty}$ also are bounded sequences.

Now, we get

$$\begin{aligned}
&\int_{-1}^{1} \hat{B}_n(x) q_n(x) w_{k,1}(x)\, d\mu(x) \\
&= \sum_{i=0}^{N} \alpha_{n,i} \int_{-1}^{1} w_{i,2}(x) q_n(x) w_{k,1}(x)\, d\mu(x) + \sum_{i=1}^{n-N} \beta_{n,i} \int_{-1}^{1} w_N(x) x^i q_n(x) w_{k,1}(x)\, d\mu(x) \\
&= \sum_{i=0}^{N-k-1} \alpha_{n,i} \int_{-1}^{1} w_{i,2}(x) w_{k,1}(x) q_n(x)\, d\mu(x)
\end{aligned}$$

from the orthogonality of $q_n(x)$ and because, for $N - k \leqslant i \leqslant N$, $w_{i,2}(x) w_{k,1}(x) = w_N(x)\pi(x)$, where $\pi(x)$ is a polynomial of degree less than or equal to k. Taking limits,

$$\begin{aligned}
\lim_{n\to\infty} \int_{-1}^{1} \hat{B}_n(x) q_n(x) w_{k,1}(x)\, d\mu(x) &= \lim_{n\to\infty} \sum_{i=0}^{N-k-1} \alpha_{n,i} \int_{-1}^{1} w_{i,2}(x) w_{k,1}(x) q_n(x)\, d\mu(x) \\
&= \lim_{n\to\infty} \sum_{i=0}^{N-k-1} \alpha_{n,i} \int_{-1}^{1} \frac{w_{i,2}(x) w_{k,1}(x)}{w_N(x)} \\
&\quad \times q_n(x) w_N(x)\, d\mu(x) = 0,
\end{aligned}$$

because $(\alpha_{n,i})_{n=0}^{\infty}$ are bounded sequences and $\int_{-1}^{1} \frac{w_{i,2}(x) w_{k,1}(x)}{w_N(x)} q_n(x) w_N(x)\, d\mu(x)$ are the Fourier coefficients of the continuous function $\frac{w_{i,2}(x) w_{k,1}(x)}{w_N(x)}$ which tend to zero. Hence, Lemma 2.1 is proved. ∎

The computation of the coefficients $b_k,\ k = 1, \ldots, N$, is straightforward. Let us consider the function

$$F(z) = \frac{1}{\pi} \int_{-1}^{1} \frac{\Pi_N(x)}{x - z} \frac{dx}{\sqrt{1 - x^2}}.$$

From the orthogonality of $\Pi_N(x)$, we have

$$0 = \frac{1}{\pi} \int_{-1}^{1} \Pi_N(x) \frac{w_N(x)}{(x - a_k)^i} \frac{dx}{w_N(x)\sqrt{1 - x^2}}, \qquad i = 1, \ldots, N_k + 1, \quad k = 1, \ldots, K.$$

Then,

$$F^{(i)}(a_k) = 0, \qquad i = 0, \ldots, N_k, \quad k = 1, \ldots, K. \tag{3}$$

Since $F(z) = \frac{-1}{\sqrt{z^2-1}}(1 + \sum_{k=1}^{N} b_k(\varphi^-(z))^k)$, where $\varphi^-(z) = z - \sqrt{z^2 - 1}$ with the square root taken in such a way that $|\varphi^-(z)| < 1$ for $z \notin [-1, 1]$, (3) means that $1 + \sum_{k=1}^{K} b_k(\varphi^-(z))^k$ vanishes at a_k with multiplicity $N_k + 1$ for $k = 1, \ldots, K$. As a consequence, taking into account that the function $w = \varphi^-(z)$ is a conformal mapping from $\bar{\mathbb{C}} \backslash [-1, 1]$ to $|w| < 1$,

$$1 + \sum_{k=1}^{N} b_k(\varphi^-(z))^k = \frac{1}{\prod_{k=1}^{K} (-\varphi^-(a_k))^{N_k+1}} \prod_{k=1}^{K} (\varphi^-(z) - \varphi^-(a_k))^{N_k+1}$$

and from the relations between A_k and b_k given in Lemma 2.1,

$$\sum_{k=0}^{N} A_k(\varphi^-(z))^k = \frac{1}{\sqrt{2^N \prod_{k=1}^{K} (-\varphi^-(a_k))^{N_k+1}}} \prod_{k=1}^{K} (\varphi^-(z) - \varphi^-(a_k))^{N_k+1}.$$

Note that if some points a_k belong to $(1, \infty)$ in such a way that $\kappa(w_N) = -1$, in (2) we would have $A_{n,N} = -\frac{\kappa(q_{n-N})}{\kappa(q_n)} \frac{1}{A_{n,0}}$ which gives $A_i = \frac{b_i}{\sqrt{-2^N b_N}}$ in Lemma 2.1. It yields

$$\sum_{k=0}^{N} A_k(\varphi^-(z))^k = \frac{1}{\sqrt{2^N \prod_{k=1}^{K} |\varphi^-(a_k)|^{N_k+1}}} \prod_{k=1}^{K} (\varphi^-(z) - \varphi^-(a_k))^{N_k+1}$$

for the general case.

As a straightforward consequence, one obtains the strong (resp. ratio) asymptotics for the polynomials $\hat{B}_n(x)$ provided that μ belongs to Szegö (resp. Nevai) class. As it was previously mentioned, these results were also obtained by López, et al. [2] and Marcellán and Van Assche [5].

COROLLARY 2.1. *If* $\mu'(x) > 0$ *a.e.* $x \in [-1, 1]$, *then*
(i)

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{q_n(x)} = \frac{1}{\sqrt{2^N \prod_{k=1}^K |\varphi^-(a_k)|^{N_k+1}}} \prod_{k=1}^K (\varphi^-(x) - \varphi^-(a_k))^{N_k+1}$$

uniformly on compact sets of $\mathbb{C}\backslash[-1, 1]$, *where* $\varphi^-(x) = x - \sqrt{x^2 - 1}$.
(ii) $n - N$ *zeros of* $\hat{B}_n(x)$ *are in* $[-1, 1]$ *and, for* $k = 1, \ldots, K$, $N_k + 1$ *zeros tend to* a_k.
(iii)

$$\lim_{n\to\infty} \frac{\hat{B}_{n+1}(x)}{\hat{B}_n(x)} = x + \sqrt{x^2 - 1}$$

uniformly on compact sets of $\mathbb{C}\backslash([-1, 1] \bigcup_{k=1}^K \{a_k\})$.
(iv) *If* $\int_{-1}^1 \log \mu'(x) \frac{dx}{\sqrt{1-x^2}} > -\infty$, *then*

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{(x + \sqrt{x^2 - 1})^n} = \frac{1}{\sqrt{2^N \prod_{k=1}^K |\varphi^-(a_k)|^{N_k+1}}} \prod_{k=1}^K (\varphi^-(x) - \varphi^-(a_k))^{N_k+1} S(x)$$

uniformly on compact sets of $\mathbb{C}\backslash[-1, 1]$, *where* $S(x)$ *is the Szegö function of* $w_N(x)\mu'(x)$ (*see* [9, *Theorem* 12.1.2] *as well as the definition in p.* 276).

Item (ii) is a consequence of the fact that $\int_{-1}^1 \hat{B}_n(x) w_N(x) x^k \, d\mu(x)$ is equal to zero for $k = 0, 1, \ldots, n - N - 1$ as well as from the asymptotic formula (i).

If we write the polynomials $w_N(x)\hat{B}_n(x)$ in terms of $\hat{B}_j(x)$ for $j = 0, \ldots, n + N$, taking into account that

$$\langle w_N(x)\hat{B}_n(x), \hat{B}_j(x)\rangle = \langle \hat{B}_n(x), w_N(x)\hat{B}_j(x)\rangle = \int_{-1}^1 \hat{B}_n(x)\hat{B}_j(x) w_N(x) \, d\mu(x),$$

which, in turn, is zero for $j = 0, \ldots, n - N - 1$, we have $w_N(x)\hat{B}_n(x) = \sum_{j=-N}^N \alpha_{n,j} \hat{B}_{n+j}(x)$ and, consequently, they satisfy a $2N + 1$ recurrence relation. Since $\alpha_{n,-j} = \alpha_{n-j,j}$, the recurrence relation can be written as

$$w_N(x)\hat{B}_n(x) = \sum_{j=0}^N \alpha_{n,j} \hat{B}_{n+j}(x) + \sum_{j=1}^N \alpha_{n-j,j} \hat{B}_{n-j}(x).$$

Besides, for $0 \leqslant j \leqslant N$,

$$\begin{aligned}\alpha_{n,j} &= \int_{-1}^{1} \hat{B}_n(x)\hat{B}_{n+j}(x)w_N(x)\,d\mu(x)\\ &= \sum_{i=0}^{N}\sum_{k=0}^{N} A_{n,i}A_{n+j,k}\int_{-1}^{1} q_{n-i}(x)q_{n+j-k}(x)w_N(x)\,d\mu(x),\end{aligned}$$

and, if $\mu'(x) > 0$ a.e., Lemma 2.1 gives

$$\lim_{n\to\infty} \alpha_{n,j} = \sum_{i=0}^{N}\sum_{k=0}^{N} A_iA_k \frac{1}{\pi}\int_{-1}^{1} T_{|i-k+j|}(x)\frac{dx}{\sqrt{1-x^2}} = \alpha_j.$$

Corollary 2.2. *There are constants $\alpha_{n,k}$ such that*

$$w_N(x)\hat{B}_n(x) = \sum_{k=0}^{N} \alpha_{n,k}\hat{B}_{n+k}(x) + \sum_{k=1}^{N} \alpha_{n-k,k}\hat{B}_{n-k}(x).$$

Moreover, if $\mu'(x) > 0$ a.e., there exist real numbers α_k such that $\lim_{n\to\infty}\alpha_{n,k} = \alpha_k$ *for* $k = 0, \ldots, N$.

In the case of only one point a_k *and* $N_k = 1$, explicit values of α_k are given in [3] and, in the general case, the values α_k can be seen in [2, 5]. For our purpose in this paper, we only need to know that the sequences $(\alpha_{n,k})_{n=0}^{\infty}$ are convergent. From Lemma 2.1, it is possible to obtain the weak asymptotic formula

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\hat{B}_n(x)\hat{B}_{n+k}(x)\,d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} f(x)T_k(x)\frac{dx}{\sqrt{1-x^2}}$$

for any continuous function $f(x)$. This formula was proved in [2, 5].

In order to study the behaviour of the Fourier–Sobolev series, we need one more result. Let us consider the already defined space $\mathbb{S}$ and let Φ be the family of polynomials

$$\Phi = \left\{\frac{w_N(x)}{x-a_1}, \ldots, \frac{w_N(x)}{(x-a_1)^{N_1+1}}, \frac{w_N(x)}{(x-a_2)}, \ldots, \frac{w_N(x)}{(x-a_K)^{N_K+1}}, w_N(x), xw_N(x), \ldots\right\}.$$

Lemma 2.2. *$\mathbb{S}$ is a Hilbert space and the family of polynomials Φ is maximal in $\mathbb{S}$.*

Proof. Since $\|f(x)\|_{\mathbb{S}}^2 = \langle f(x), f(x)\rangle = \|f(x)\|_{\mu}^2 + \sum_{k=1}^{K}\sum_{i=0}^{N_k} M_{k,i}|f^{(i)}(a_k)|^2$, a Cauchy sequence in $\mathbb{S}, (f_n)_{n=0}^{\infty}$, is a Cauchy sequence in $L^2(\mu)$ and the same happens for the sequences $(f_n^{(i)}(a_k))_{n=0}^{\infty}$ in $\mathbb{R}$. Then, any function f, defined in a neighbourhood of a_k such that $f^{(i)}(a_k) = \lim_{n\to\infty} f_n^{(i)}(a_k)$ and that in $[-1,1]$ is the $L^2(\mu)$ limit of f_n, is a limit of f_n in $\mathbb{S}$. Therefore, $\mathbb{S}$ is a Hilbert space.

On the other hand, if $\langle f(x), x^k w_N(x)\rangle = 0$ for $k = 0, 1, \ldots,$ then

$$\int_{-1}^{1} f(x)x^k w_N(x)\, d\mu(x) = 0, \qquad k = 0, 1, \ldots$$

and thus $w_N(x)f(x) = 0$ μ-a.e. But $w_N(x) > 0$ for $x \in [-1,1]$, hence $f(x) = 0$ μ-a.e. In this case, $\langle f(x), g(x)\rangle = \sum_{k=1}^{K} \sum_{i=1}^{N_k} M_{k,i} f^{(i)}(a_k) g^{(i)}(a_k)$ and, from $\langle f(x), \frac{w_N(x)}{(x-a_k)^i}\rangle = 0$, $f^{(N_k+1-i)}(a_k) = 0$ for $i = 1, \ldots, N_k + 1$ and $k = 1, \ldots, K$. As a consequence, $f = 0$ in $\mathbb{S}$ and the lemma is proved. ∎

3. ESTIMATES FOR SOBOLEV POLYNOMIALS

In order to obtain estimates for $\hat{B}_n(x)$ when $x \in [-1,1]$, the measure μ is considered to be in the Nevai class.

LEMMA 3.1. *Let μ be a measure such that $\mu'(x) > 0$ a.e. $x \in [-1,1]$. Let $(p_n(x))_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to μ. Let $a \in \mathbb{R}\backslash[-1,1]$ and let $(t_n(x))_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to $|x-a|\, d\mu(x)$. There exists a positive constant C such that*

$$|x-a||t_n(x)| \leqslant C(|p_{n+1}(x)| + |p_n(x)|)$$

for every x and for all n.

Proof. For the polynomials $t_n(x)$ we have $t_n(x) = \sum_{j=0}^{n} \lambda_{n,j} p_j(x)$, where

$$\lambda_{n,j} = \int_{-1}^{1} t_n(s) p_j(s)\, d\mu(s) = p_j(a) \int_{-1}^{1} t_n(s)\, d\mu(s)$$

$$+ \int_{-1}^{1} t_n(s)(s-a) \sum_{k=1}^{j} \frac{p_j^{(k)}(a)}{k!}(s-a)^{k-1}\, d\mu(s) = p_j(a) \int_{-1}^{1} t_n(s)\, d\mu(s).$$

Hence,

$$\begin{aligned} t_n(x) &= \int_{-1}^{1} t_n(s)\, d\mu(s) \sum_{j=0}^{n} p_j(a) p_j(x) \\ &= \int_{-1}^{1} t_n(s)\, d\mu(s) \frac{\kappa(p_n)}{\kappa(p_{n+1})} \frac{p_{n+1}(x)p_n(a) - p_{n+1}(a)p_n(x)}{x-a}, \end{aligned}$$

and, as a consequence,

$$|t_n(x)| \leqslant \frac{\kappa(p_n)}{\kappa(p_{n+1})} \left| \int_{-1}^{1} t_n(s)\, p_n(a)\, d\mu(s) \right| \frac{1}{|x-a|} \left(|p_{n+1}(x)| + \frac{|p_{n+1}(a)|}{|p_n(a)|} |p_n(x)| \right).$$

But $\frac{\kappa(p_n)}{\kappa(p_{n+1})}$ and $\frac{p_{n+1}(a)}{p_n(a)}$ are bounded because these polynomials have ratio asymptotics. Thus,

$$\left| \int_{-1}^{1} t_n(s)\, p_n(a)\, d\mu(s) \right| = \left| \int_{-1}^{1} t_n(s)\, p_n(s)\, d\mu(s) \right| \leqslant \left(\int_{-1}^{1} t_n^2(s)\, d\mu(s)| \right)^{1/2}$$
$$\leqslant \frac{1}{\sqrt{dist(a, [-1,1])}}.$$

Then, $|x-a||t_n(x)| \leqslant C(|p_{n+1}(x)| + |p_n(x)|)$ for every x and for some constant C. ∎

By iteration of this lemma, for the polynomials $(q_n(x))_{n=0}^{\infty}$, orthonormal with respect to $w_N(x)\, d\mu(x)$, we get

Corollary 3.1. *If $\mu'(x) > 0$ a.e. and $p_n(x)$ are orthonormal with respect to μ, there exists a positive constant C such that*

$$|w_N(x)||q_n(x)| \leqslant C(|p_{n+N}(x)| + \cdots + |p_n(x)|)$$

for every x and for all n.

For the Sobolev orthonormal polynomials $(\hat{B}_n(x))_{n=0}^{\infty}$, this inequality and Lemma 2.1 give

Corollary 3.2. *If $\mu'(x) > 0$ a.e. and $p_n(x)$ are orthonormal with respect to μ, there exists a positive constant C such that*

$$|w_N(x)||\hat{B}_n(x)| \leqslant C(|p_{n+N}(x)| + \cdots + |p_{n-N}(x)|)$$

for every x and for all n.

Corollary 3.3. *If $\mu'(x) > 0$ a.e. and there is a function $h(x)$ such that the polynomials $p_n(x)$, orthonormal with respect to μ, satisfy the condition $|p_n(x)| \leqslant h(x)$, $x \in [-1,1]$, then there exists a constant C such that*

$$|\hat{B}_n(x)| \leqslant C h(x)$$

for $x \in [-1,1]$ and for all n.

It is clear that the constants C in the previous corollaries may be different despite the fact that we use the same symbol.

The last corollary will be very useful for the study of Fourier–Sobolev series when μ is the Jacobi measure because in this case the function $h(x)$ is very well known.

In order to study the Fourier series, we also need estimates of $\hat{B}_n^{(j)}(a_k)$, $j = 0, \ldots, N_k$ and $k = 1, \ldots, K$. This problem will be considered now. The condition $\mu'(x) > 0$ a.e. is not sufficient for our purposes in what follows. Thus, from now on, we will consider the measure μ in the Szegö class, i.e. $\int_{-1}^{1} \log \mu'(x) \frac{dx}{\sqrt{1-x^2}} > -\infty$.

LEMMA 3.2. *Let μ be a measure in the Szegö class and let $q_n(x)$ be the orthonormal polynomials with respect to $w_N(x)\,d\mu(x)$. There is a constant C such that*

$$\left| \int_{-1}^{1} q_n(x) \frac{w_N(x)}{(x-a_k)^i}\, d\mu(x) \right| \leqslant C \frac{n^{i-1}}{|a_k + \sqrt{a_k^2 - 1}|^n}$$

for $i = 1, \ldots, N_k + 1$, $k = 1, \ldots, K$ and n large enough.

Proof. We proceed by induction. By orthogonality,

$$\begin{aligned} \int_{-1}^{1} q_n(x) \frac{w_N(x)}{x-a_k}\, d\mu(x) &= \frac{1}{q_n(a_k)} \int_{-1}^{1} q_n(x) q_n(a_k) \frac{w_N(x)}{x-a_k}\, d\mu(x) \\ &= \frac{1}{q_n(a_k)} \int_{-1}^{1} q_n(x) \{ q_n(a_k) \\ &\quad + (x-a_k)\pi_{n-1}(x) \} \frac{w_N(x)}{x-a_k}\, d\mu(x) \end{aligned}$$

for any polynomial $\pi_{n-1}(x)$ of degree less than n. Then,

$$\left| \int_{-1}^{1} q_n(x) \frac{w_N(x)}{x-a_k}\, d\mu(x) \right| = \frac{1}{|q_n(a_k)|} \left| \int_{-1}^{1} q_n^2(x) \frac{w_N(x)}{x-a_k}\, d\mu(x) \right| \leqslant \frac{C_1}{|q_n(a_k)|}.$$

Suppose

$$\left| \int_{-1}^{1} q_n(x) \frac{w_N(x)}{(x-a_k)^j}\, d\mu(x) \right| \leqslant C_j \frac{n^{j-1}}{|q_n(a_k)|}$$

for some constant C_j and for $1 \leqslant j \leqslant i \leqslant N_k$. Then,

$$\begin{aligned}
&\int_{-1}^{1} q_n(x) \frac{w_N(x)}{(x-a_k)^{i+1}}\, d\mu(x) \\
&= \frac{1}{q_n(a_k)} \int_{-1}^{1} q_n(x) \left\{ q_n(a_k) + \sum_{v=1}^{i} \frac{q_n^{(v)}(a_k)}{v!}(x-a_k)^v \right\} \frac{w_N(x)}{(x-a_k)^{i+1}}\, d\mu(x) \\
&\quad - \frac{1}{q_n(a_k)} \int_{-1}^{1} q_n(x) \sum_{v=1}^{i} \frac{q_n^{(v)}(a_k)}{v!}(x-a_k)^v \frac{w_N(x)}{(x-a_k)^{i+1}}\, d\mu(x) \\
&= \frac{1}{q_n(a_k)} \int_{-1}^{1} q_n^2(x) \frac{w_N(x)}{(x-a_k)^{i+1}}\, d\mu(x) - \frac{1}{q_n(a_k)} \sum_{v=1}^{i} \frac{q_n^{(v)}(a_k)}{v!} \\
&\quad \times \int_{-1}^{1} q_n(x) \frac{w_N(x)}{(x-a_k)^{i+1-v}}\, d\mu(x).
\end{aligned}$$

By induction,

$$\left| \int_{-1}^{1} q_n(x) \frac{w_N(x)}{(x-a_k)^{i+1}}\, d\mu(x) \right| \leqslant \frac{C^*}{|q_n(a_k)|} + \sum_{v=1}^{i} \frac{|q_n^{(v)}(a_k)|}{|v! q_n(a_k)|} \frac{C_{i+1-v} n^{i-v}}{|q_n(a_k)|},$$

but, since μ belongs to the Szegö class, $\frac{|q_n^{(v)}(a_k)|}{|q_n(a_k)|} \leqslant C^{**} n^v$, and

$$\left| \int_{-1}^{1} q_n(x) \frac{w_N(x)}{(x-a_k)^{i+1}}\, d\mu(x) \right| \leqslant \frac{C^*}{|q_n(a_k)|} + \frac{C^{**} n^i}{|q_n(a_k)|} \sum_{v=1}^{i} \frac{C_{i+1-v}}{v!} \leqslant \frac{C_{i+1} n^i}{|q_n(a_k)|}$$

for some constant C_{i+1} and n large enough. ∎

COROLLARY 3.4. *If μ belongs to the Szegö class, there is a constant C such that*

$$|\hat{B}_n^{(i)}(a_k)| \leqslant C \frac{n^{N_k - i}}{|a_k + \sqrt{a_k^2 - 1}|^n}$$

for $i = 0, \ldots, N_k$, $k = 1, \ldots, K$ and n large enough.

Proof. We use induction again. Since

$$\begin{aligned}
0 &= \left\langle \hat{B}_n(x), \frac{w_N(x)}{x-a_k} \right\rangle \\
&= \int_{-1}^{1} \hat{B}_n(x) \frac{w_N(x)}{x-a_k}\, d\mu(x) + M_{k,N_k} \hat{B}_n^{(N_k)}(a_k) \frac{w_N^{(N_k+1)}(a_k)}{N_k+1},
\end{aligned}$$

we get

$$|\hat{B}_n^{(N_k)}(a_k)| = \frac{N_k+1}{M_{k,N_k}|w_N^{(N_k+1)}(a_k)|}\left|\int_{-1}^{1}\hat{B}_n(x)\frac{w_N(x)}{x-a_k}\,d\mu(x)\right|.$$

Hence, Lemmas 2.1 and 3.2 give $\hat{B}_n^{(N_k)}(a_k) = O(\frac{1}{(a_k+\sqrt{a_k^2-1})^n})$.

We assume $\hat{B}_n^{(N_k-j)}(a_k) = O(\frac{n^{N_k-j}}{(a_k+\sqrt{a_k^2-1})^n})$ for $0\leqslant j\leqslant i<N_k$. Then, we have

$$0 = \left\langle \hat{B}_n(x), \frac{w_N(x)}{(x-a_k)^{N_k-i}}\right\rangle = \int_{-1}^{1}\hat{B}_n(x)\,\frac{w_N(x)}{(x-a_k)^{N_k-i}}\,d\mu(x)$$

$$+\ M_{k,i+1}\hat{B}_n^{(i+1)}(a_k)\,\frac{(i+1)!w_N^{(N_k+1)}(a_k)}{(N_k+1)!} + O\left(\frac{n^{N_k-(i+2)}}{(a_k+\sqrt{a_k^2-1})^n}\right),$$

whence

$$\hat{B}_n^{(i+1)}(a_k) = \frac{-(N_k+1)!}{M_{k,i+1}(i+1)!w_N^{(N_k+1)}(a_k)}\int_{-1}^{1}\hat{B}_n(x)\frac{w_N(x)}{(x-a_k)^{N_k-i}}\,d\mu(x)$$

$$+\ O\left(\frac{n^{N_k-(i+2)}}{(a_k+\sqrt{a_k^2-1})^n}\right).$$

But, from Lemmas 2.1 and 3.2,

$$\int_{-1}^{1}\hat{B}_n(x)\,\frac{w_N(x)}{(x-a_k)^{N_k-i}}\,d\mu(x) = O\left(\frac{n^{N_k-(i+1)}}{(a_k+\sqrt{a_k^2-1})^n}\right).$$

Then $\hat{B}_n^{(i+1)}(a_k) = O(\frac{n^{N_k-(i+1)}}{(a_k+\sqrt{a_k^2-1})^n})$. This completes the proof. ∎

4. FOURIER SERIES

In Lemma 2.2 we proved that, with the inner product (1),

$$\mathbb{S}=\left\{f(x):\int_{-1}^{1}|f(x)|^2\,d\mu(x)<\infty,\ f^{(i)}(a_k)\ \text{exists for}\ i=0,\dots,N_k,\ \ k=1,\dots,K\right\}$$

is a Hilbert space and the polynomials constitute a maximal family. Then, $S_n(f) \to f$ in $\mathbb{S}$ for any function $f \in \mathbb{S}$, where

$$S_n(x; f) = \sum_{k=0}^{n} \langle f, \hat{B}_k \rangle \hat{B}_k(x)$$

is the nth partial sum of the Fourier–Sobolev series of f. Write $\|f\|_{\mathbb{S}}^2 = \|f\|_\mu^2 + \|f\|_d^2$. Convergence in $\mathbb{S}$ induces convergence in $L^2(\mu)$ as well as convergence for the derivatives at the points a_k because $\|f\|_\mu^2 \leqslant \|f\|_{\mathbb{S}}^2$ and $\|f\|_d^2 \leqslant \|f\|_{\mathbb{S}}^2$. So, for any function f in $\mathbb{S}$, we have

$$S_n(x; f) \overset{L^2(\mu)}{\to} f(x), \qquad S_n^{(i)}(a_k; f) \to f^{(i)}(a_k), \quad 0 \leqslant i \leqslant N_k, \quad k = 1\dots, K.$$

For $i = 0, \dots, N_k$ and $k = 1, \dots, K$, let us consider the functions $f_{k,i}$ such that $f_{k,i}(x) = 0,\ x \in [-1, 1],\ f_{k,i}^{(j)}(a_t) = 1$ when $t = k,\ j = i$, and 0 otherwise. Since $S_n(f_{k,i})$ converges to $f_{k,i}$ in $\mathbb{S}$ and $\langle f_{k,i}, \hat{B}_n \rangle = M_{k,i} \hat{B}_n^{(i)}(a_k)$, we get

$$\sum_{n=0}^{\infty} \hat{B}_n^{(i)}(a_k) \hat{B}_n(x) \overset{L^2(\mu)}{=} 0, \qquad \sum_{n=0}^{\infty} \hat{B}_n^{(i)}(a_k) \hat{B}_n^{(j)}(a_t) = 0, \quad t \neq k \quad \text{or} \quad j \neq i,$$

$$\sum_{n=0}^{\infty} (\hat{B}_n^{(i)}(a_k))^2 = \frac{1}{M_{k,i}}.$$

Let μ be the Jacobi measure, $d\mu(x) = (1-x)^\alpha (1+x)^\beta\, dx,\ \alpha > -1,\ \beta > -1$, and let $p_n(x) = p_n^{(\alpha,\beta)}(x)$ the corresponding orthonormal polynomials (from now on, the orthonormal Jacobi–Sobolev polynomials). As it is well known (see [8, Theorem 3.14, p. 101]) that

$$(1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4}|p_n(x)| \leqslant C, \qquad x \in [-1, 1].$$

Let $\hat{B}_n(x) = \hat{B}_n^{(\alpha,\beta)}(x)$ be the orthonormal polynomials with respect to the inner product (1) when μ is the Jacobi measure. Corollary 3.3 yields the uniform bound

$$|\hat{B}_n(x)| \leqslant \frac{C}{(1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4}} = h(x), \qquad x \in (-1, 1). \tag{4}$$

From inequality (4) and Corollary 3.4, the series $\sum_{n=0}^{\infty} \hat{B}_n^{(i)}(a_k)\hat{B}_n(x),\ 0 \leqslant i \leqslant N_k$, has the majorant $\sum_{n=0}^{\infty} Cn^{N_k - i}(a_k - \sqrt{a_k^2 - 1})^n$ in compact sets of $(-1, 1)$ for some constant C. Then, the series is a continuous function in $(-1, 1)$. But we have convergence to 0 in $L^2(\mu)$ for the series. Hence, it has a subsequence which converges pointwise to 0 a.e. As a consequence, $\sum_{n=0}^{\infty} \hat{B}_n^{(i)}(a_k)\hat{B}_n(x) = 0$ for all $x \in (-1, 1)$. We summarize the above as follows.

THEOREM 4.1. *Let $\hat{B}_n(x)$ be the orthonormal Jacobi–Sobolev polynomials. Then,*

(i) $\sum_{n=0}^{\infty} \hat{B}_n^{(i)}(a_k)\hat{B}_n(x) = 0$ *for every* $x \in (-1, 1)$, $i = 0, \ldots, N_k$ *and* $k = 1, \ldots, K$.

(ii) $\sum_{n=0}^{\infty} \hat{B}_n^{(i)}(a_k)\hat{B}_n^{(j)}(a_t) = 0$ *for* $t \neq k$ *or* $j \neq i$.

(iii) $\sum_{n=0}^{\infty}(\hat{B}_n^{(i)}(a_k))^2 = \frac{1}{M_{k,i}}$ *for* $i = 0, \ldots, N_k$ *and* $k = 1, \ldots, K$.

From now on, we will study the pointwise convergence of $S_n(f)$ to f on the interval $[-1, 1]$ when there are standard sufficient conditions for the function f. First of all, we need the analogous of the Christoffel–Darboux formula for the Sobolev polynomials but, if $x_0 \in [-1, 1]$, the polynomial $w_N(x) - w_N(x_0)$ can have two zeros in the interval $[-1, 1]$ when there are points a_k in $(-\infty, -1)$ and in $(1, \infty)$ simultaneously. Then, this polynomial is not convenient for representing the Dirichlet kernel. Instead of $w_N(x)$, we will consider a different polynomial which also allows a Christoffel–Darboux-type formula and which has better properties. Let $w_{N+1}^*(x) = \int_0^x w_N(t)\,dt$ and let $c = \min\{w_{N+1}^*(x) : x \in [-1, 1]\}$. Let $w_{N+1}(x)$ be the polynomial $w_{N+1}^*(x) + |c| + 1$. It is clear that $w_{N+1}(x)$ does not have zeros in $[-1, 1]$ and, when $x_0 \in [-1, 1]$, $w_{N+1}(x) - w_{N+1}(x_0)$ has the only zero x_0 in $[-1, 1]$ because its derivative $w_N(x)$ does not vanish at this interval. The important facts are that $\frac{x-x_0}{w_{N+1}(x)-w_{N+1}(x_0)}$ is a continuous function in $[-1, 1]$ and that we can obtain an expression for the Dirichlet kernel in terms of $w_{N+1}(x)$. Since the derivatives of $w_{N+1}(x)$ are equal to zero at the points a_k,

$$\langle w_{N+1}(x)f(x), g(x)\rangle = \langle f(x), w_{N+1}(x)g(x)\rangle,$$

and, as a consequence, we have the following recurrence relations for the polynomials $\hat{B}_n(x)$,

$$w_{N+1}(x)\hat{B}_n(x) = \sum_{k=0}^{N+1} \alpha_{n,k}\hat{B}_{n+k}(x) + \sum_{k=1}^{N+1} \alpha_{n-k,k}\hat{B}_{n-k}(x). \tag{5}$$

Moreover, the coefficients $\alpha_{n,k}$ are bounded because

$$\begin{aligned}|\alpha_{n,k}| &= |\langle w_{N+1}\hat{B}_n, \hat{B}_{n+k}\rangle| \\ &\leqslant \left|\int_{-1}^{1} \hat{B}_n(x)\hat{B}_{n+k}(x)w_{N+1}(x)\,d\mu(x)\right| + \sum_{i=1}^{K} M_{i,0}w_{N+1}(a_i)|\hat{B}_n(a_i)||\hat{B}_{n+k}(a_i)|\end{aligned}$$

and the first term is bounded by $\max_{x\in[-1,1]}|w_{N+1}(x)|$ and the other one is also bounded from Corollary 3.4.

The Christoffel–Darboux formula takes now the following form (see [4]).

LEMMA 4.1. *Orthonormal polynomials with respect to the inner product* (1) *satisfy the following Christoffel–Darboux-type formula*:

$$
\begin{aligned}
&\{w_{N+1}(x) - w_{N+1}(y)\} \sum_{n=0}^{v} \hat{B}_n(x)\hat{B}_n(y) = \alpha_{v,1}(\hat{B}_{v+1}(x)\hat{B}_v(y) - \hat{B}_{v+1}(y)\hat{B}_v(x)) \\
&+ \alpha_{v,2}(\hat{B}_{v+2}(x)\hat{B}_v(y) - \hat{B}_{v+2}(y)\hat{B}_v(x)) + \alpha_{v-1,2}(\hat{B}_{v+1}(x)\hat{B}_{v-1}(y) \\
&- \hat{B}_{v+1}(y)\hat{B}_{v-1}(x)) + \cdots + \alpha_{v,N+1}(\hat{B}_{v+N+1}(x)\hat{B}_v(y) - \hat{B}_{v+N+1}(y)\hat{B}_v(x)) \\
&+ \cdots + \alpha_{v-N,N+1}(\hat{B}_{v+1}(x)\hat{B}_{v-N}(y) - \hat{B}_{v+1}(y)\hat{B}_{v-N}(x)).
\end{aligned}
$$

Furthermore, if the measure belongs to the Szegö class, the coefficients are bounded.

Proof. As usual, from (5) we have

$$
\begin{aligned}
w_{N+1}(x)\hat{B}_n(x)\hat{B}_n(y) &= \sum_{k=0}^{N+1} \alpha_{n,k}\hat{B}_{n+k}(x)\hat{B}_n(y) + \sum_{k=1}^{N+1} \alpha_{n-k,k}\hat{B}_{n-k}(x)\hat{B}_n(y), \\
w_{N+1}(y)\hat{B}_n(y)\hat{B}_n(x) &= \sum_{k=0}^{N+1} \alpha_{n,k}\hat{B}_{n+k}(y)\hat{B}_n(x) + \sum_{k=1}^{N+1} \alpha_{n-k,k}\hat{B}_{n-k}(y)\hat{B}_n(x).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\{w_{N+1}(x) - w_{N+1}(y)\}\hat{B}_n(x)\hat{B}_n(y) \\
&= \sum_{k=N+1}^{1} \alpha_{n,k}(\hat{B}_{n+k}(x)\hat{B}_n(y) - \hat{B}_{n+k}(y)\hat{B}_n(x)) \\
&\quad - \sum_{k=1}^{N+1} \alpha_{n-k,k}(\hat{B}_n(x)\hat{B}_{n-k}(y) - \hat{B}_n(y)\hat{B}_{n-k}(x)).
\end{aligned}
$$

Writing $F_n^k(x, y) = \alpha_n^k(\hat{B}_{n+k}(x)\hat{B}_n(y) - \hat{B}_{n+k}(y)\hat{B}_n(x))$ and taking into account that $F_n^k(x, y) = 0$ for negative integer values of n, we get

$$
\begin{aligned}
&\{w_{N+1}(x) - w_{N+1}(y)\} \sum_{n=0}^{v} \hat{B}_n(x)\hat{B}_n(y) \\
&= \sum_{n=0}^{v} \{(F_n^1(x, y) - F_{n-1}^1(x, y)) + (F_n^2(x, y) - F_{n-2}^2(x, y)) + \cdots + (F_n^{N+1}(x, y) \\
&\quad - F_{n-N-1}^{N+1}(x, y))\} \\
&= F_v^1(x, y) + F_v^2(x, y) + F_{v-1}^2(x, y) + F_v^3(x, y) + F_{v-1}^3(x, y) + F_{v-2}^3(x, y) + \cdots \\
&\quad + F_v^{N+1}(x, y) + \cdots + F_{v-N}^{N+1}(x, y)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_{\nu,1}(\hat{B}_{\nu+1}(x)\hat{B}_\nu(y) - \hat{B}_{\nu+1}(y)\hat{B}_\nu(x)) + \alpha_{\nu,2}(\hat{B}_{\nu+2}(x)\hat{B}_\nu(y) - \hat{B}_{\nu+2}(y)\hat{B}_\nu(x)) \\
&\quad + \alpha_{\nu-1,2}(\hat{B}_{\nu+1}(x)\hat{B}_{\nu-1}(y) - \hat{B}_{\nu+1}(y)\hat{B}_{\nu-1}(x)) + \cdots \\
&\quad + \alpha_{\nu,N+1}(\hat{B}_{\nu+N+1}(x)\hat{B}_\nu(y) - \hat{B}_{\nu+N+1}(y)\hat{B}_\nu(x)) + \cdots \\
&\quad + \alpha_{\nu-N,N+1}(\hat{B}_{\nu+1}(x)\hat{B}_{\nu-N}(y) - \hat{B}_{\nu+1}(y)\hat{B}_{\nu-N}(x)).
\end{aligned}
$$

THEOREM 4.2. *Let $x_0 \in (-1, 1)$ and let f be a function with derivatives at the points a_k such that $\frac{f(x_0)-f(t)}{x_0-t}$ belongs to $L^2(\mu)$ where μ is the Jacobi measure. Then,*

(i) $\sum_{n=0}^{\infty}\langle f, \hat{B}_n\rangle \hat{B}_n(x_0) = f(x_0)$.

(ii) $\sum_{n=0}^{\infty}\langle f, \hat{B}_n\rangle \hat{B}_n^{(i)}(a_k) = f^{(i)}(a_k), \quad i = 0, \ldots, N_k, \;\; k = 1, \ldots, K.$

Proof. Since $f \in L^2(\mu)$ provided that $\frac{f(x_0)-f(t)}{x_0-t} \in L^2(\mu)$, (ii) is proved. Thus, we only need to prove (i). Let us denote $D_n(x,t) = \sum_{j=0}^{n} \hat{B}_j(x)\hat{B}_j(t)$. We have

$$
\begin{aligned}
f(x_0) - S_n(x_0; f) &= \langle f(x_0) - f(t), D_n(x_0, t)\rangle \\
&= \int_{-1}^{1} (f(x_0) - f(t)) D_n(x_0, t)\, d\mu(t) \\
&\quad + \sum_{k=1}^{K} M_{k,0}(f(x_0) - f(a_k)) D_n(x_o, a_k) \\
&\quad - \sum_{k=1}^{K}\sum_{i=1}^{N_k} M_{k,i} f^{(i)}(a_k) \frac{\partial^i D_n}{\partial t^i}(x_0, a_k).
\end{aligned}
$$

But Theorem 4.1 yields $\lim_{n\to\infty} D_n(x_0, a_k) = \lim_{n\to\infty} \frac{\partial^i D_n}{\partial t^i}(x_0, a_k) = 0$ for $i = 1, \ldots, N_k, \;\; k = 1, \ldots, K$. Then,

$$
\lim_{n\to\infty} (f(x_0) - S_n(x_0; f)) = \lim_{n\to\infty} \int_{-1}^{1} (f(x_0) - f(t)) D_n(x_0, t)\, d\mu(t).
$$

From the Christoffel–Darboux formula (Lemma 4.1), $D_n(x_0, t)$ is a sum of a finite number of terms—depending on N—of the following type:

$$
\alpha_{n-i,j} \frac{\hat{B}_{n-i+j}(x_0)\hat{B}_{n-i}(t)}{w_{N+1}(x_0) - w_{N+1}(t)}, \quad 0 \leqslant i \leqslant N, \quad 1 \leqslant j \leqslant N+1.
$$

Taking into account that

$$\left|\int_{-1}^{1}(f(x_0)-f(t))\alpha_{n-i,j}\frac{\hat{B}_{n-i+j}(x_0)\hat{B}_{n-i}(t)}{w_{N+1}(x_0)-w_{N+1}(t)}\,d\mu(t)\right|$$
$$=|\alpha_{n-i,j}||\hat{B}_{n-i+j}(x_0)|\left|\int_{-1}^{1}\frac{f(x_0)-f(t)}{x_0-t}\frac{x_0-t}{w_{N+1}(x_0)-w_{N+1}(t)}\hat{B}_{n-i}(t)\,d\mu(t)\right|$$

as well as $|\hat{B}_{n-i+j}(x_0)|\leqslant h(x_0)$ from (4), and that $\alpha_{n-i,j}$ are bounded from Lemma 4.1, since $\frac{f(x_0)-f(t)}{x_0-t}\frac{x_0-t}{w_{N+1}(x_0)-w_{N+1}(t)}$ belongs to $L^2(\mu)$ because $\frac{x_0-t}{w_{N+1}(x_0)-w_{N+1}(t)}$ is a continuous function at $[-1,1]$ and, by hypothesis, $\frac{f(x_0)-f(t)}{x_0-t}$ belongs to $L^2(\mu)$, Lemma 2.1 gives

$$\lim_{n\to\infty}\alpha_{n-i,j}\hat{B}_{n-i+j}(x_0)\int_{-1}^{1}\frac{f(x_0)-f(t)}{x_0-t}\frac{x_0-t}{w_{N+1}(x_0)-w_{N+1}(t)}$$
$$\hat{B}_{n-i}(t)\frac{w_N(t)}{w_N(t)}\,d\mu(t)=0.$$

Hence, $\lim_{n\to\infty}(f(x_0)-S_n(x_0;f))=0$ and the proof is complete. ∎

THEOREM 4.3. *Let $f(x)$ be a function with derivatives at the points a_k satisfying a Lipschitz condition of order $\eta<1$ uniformly in $[-1,1]$, i.e. $|f(x+h)-f(x)|\leqslant M|h|^\eta$ for $|h|<\delta$ and for some $\delta>0$. If $c_n=\langle f,\hat{B}_n\rangle$, then*

$$\sum_{n=0}^{\infty}c_n\hat{B}_n(x)=f(x),\quad x\in(-1,1),$$

and the convergence is uniform in $[-1+\varepsilon,1-\varepsilon]$ for every ε such that $0<\varepsilon<1$. Moreover, $\sum_{n=0}^{\infty}c_n\hat{B}_n^{(i)}(a_k)=f^{(i)}(a_k)$ for $i=0,\dots,N_k$ and $k=1,\dots,K$.

Proof. In the same way as before, we only need to prove that $\int_{-1}^{1}f(t)D_n(x,t)\,d\mu(t)$ converges to $f(x)$ for $x\in(-1,1)$. Besides,

$$\left|\int_{-1}^{1}(f(x)-f(t))D_n(x,t)\,d\mu(t)\right|$$
$$\leqslant\left|\int_{|x-t|\geqslant\delta}(f(x)-f(t))D_n(x,t)\,d\mu(t)\right|+\left|\int_{|x-t|<\delta}(f(x)-f(t))D_n(x,t)\,d\mu(t)\right|$$
$$=I_n^{(1)}(x)+I_n^{(2)}(x).$$

Since $\frac{f(x)-f(t)}{w_{N+1}(x)-w_{N+1}(t)}(1-\chi_{(x-\delta,x+\delta)}(t))$, where $\chi_{(x-\delta,x+\delta)}(t)$ is the characteristic function of the interval, belongs to $L^2(\mu)$, using Christoffel–Darboux formula and the same procedure as in the previous Theorem, the term $I_n^{(1)}(x)$ tends to zero.

On the other hand, $I_n^{(2)}(x)$ is a sum of a finite number of terms

$$\alpha_{n-i,j}\hat{\boldsymbol{B}}_{n-i+j}(x) \int_{|x-t|<\delta} \frac{f(x)-f(t)}{x-t} \frac{x-t}{w_{N+1}(x)-w_{N+1}(t)} \hat{\boldsymbol{B}}_{n-i}(t)\, d\mu(t),$$

where the coefficients $\alpha_{n-i,j}\hat{\boldsymbol{B}}_{n-i+j}(x)$ are uniformly bounded in closed subsets of $(-1,1)$ from Lemma 4.1 and (4). Furthermore, when x belongs to $(-1,1)$, the Lipschitz condition gives

$$\left| \int_{|x-t|<\delta} \frac{f(x)-f(t)}{x-t} \frac{x-t}{w_{N+1}(x)-w_{N+1}(t)} \hat{\boldsymbol{B}}_{n-i}(t)\, d\mu(t) \right| \leqslant C \int_{|x-t|<\delta} \frac{d\mu(t)}{|x-t|^{1-\eta}},$$

where the constant C depends on $\max\{\frac{|x-t|}{|w_{N+1}(x)-w_N(t)|} : t \in [-1,1]\}$, on the constant of the Lipschitz condition and on $h(x)$, where $h(x)$ is the function such that $|\hat{\boldsymbol{B}}_n(x)| \leqslant h(x)$ on the interval $(-1,1)$. Hence, since μ is the Jacobi measure, for $\varepsilon > 0$ there exists $\delta > 0$ such that $|I_n^{(2)}| < \varepsilon$ and the pointwise convergence is proved. The uniform convergence in a compact subset F of $(-1,1)$ is an easy consequence of the uniform continuity of $\frac{f(y)-f(t)}{w_{N+1}(y)-w_{N+1}(t)}$ when (y,t) belong to $\{(y,t) : |y-t| \leqslant \frac{\delta}{2},\ |t-x| \geqslant \delta,\ x, y \in F\}$ for a fixed $x \in F$ and for a fixed δ such that $\int_{|x-t|<\delta} \frac{d\mu(t)}{|x-t|^{1-\eta}} < \varepsilon$, and of the compactness of F. ∎

As usual, denote

$$w(\delta) = w(\delta, f) = \sup\{|f(x_1) - f(x_2)| : x_1,\ x_2 \in [-1,1],\ |x_1 - x_2| \leqslant \delta\},$$

the modulus of continuity of a function $f(x)$ in $[-1,1]$.

THEOREM 4.4. *Let $f(x)$ be a function such that its modulus of continuity $w(\delta)$ satisfies the condition*

$$w(\delta) = O\left(\log^{-(1+\varepsilon)} \frac{1}{\delta}\right)$$

for $\varepsilon > 0$ with derivatives at the points a_k. If $c_n = \langle f, \hat{\boldsymbol{B}}_n \rangle$, then $\sum_{n=0}^{\infty} c_n \hat{\boldsymbol{B}}_n(x) = f(x)$ a.e. in $[-1,1]$. Moreover, $\sum_{n=0}^{\infty} c_n \hat{\boldsymbol{B}}_n^{(i)}(a_k) = f^{(i)}(a_k)$ for $i = 1, \ldots, N_k$ and $k = 1, \ldots, K$.

Proof. Note that $\sum_{n=0}^{\infty} c_n \hat{\boldsymbol{B}}_n^{(i)}(a_k) = f^{(i)}(a_k)$ holds because $f(x)$ belongs to $\mathbb{S}$ and the only thing to prove is the a.e. convergence in $[-1,1]$.

We consider again the polynomial $w_N(x)$ and the orthonormal polynomials $q_n(x)$ with respect to $w_N(x)\, d\mu(x)$. Since $w_N(x)$ has no zeros in $[-1,1]$, the modulus of continuity of $\frac{f(x)}{w_N(x)}$ satisfies the condition $w(\delta, \frac{f(x)}{w_N(x)}) = O(\log^{-(1+\varepsilon)} \frac{1}{\delta})$.

Let $d_n = \int_{-1}^{1} f(x) q_n(x)\, d\mu(x)$ be the Fourier coefficients of $\frac{f(x)}{w_N(x)}$ in terms of $q_n(x)$. By Jackson's Approximation Theorem (see [8, Chapt. I]), there is a polynomial $\pi_n(x)$ such that $|\frac{f(x)}{w_N(x)} - \pi_n(x)| = O\left(\frac{1}{\log^{1+\varepsilon} n}\right)$. Hence,

$$\sum_{k=n}^{\infty} d_k^2 = \int_{-1}^{1} \left(\frac{f(x)}{w_N(x)} - \pi_n(x)\right)^2 w_N(x)\, d\mu(x) = O\left(\frac{1}{\log^{2+2\varepsilon} n}\right). \tag{6}$$

From Lemma 2.1,

$$c_n = \langle f, \hat{B}_n \rangle = \sum_{i=0}^{N} A_{n,i} d_{n-i} + \sum_{k=1}^{K} \sum_{j=0}^{N_k} M_{k,j} f^{(j)}(a_k) \hat{B}_n^{(j)}(a_k).$$

From the bounds of Corollary 3.4 and taking into account the Cauchy–Schwarz inequality, i.e. $|\sum_{k=n}^{\infty} d_k d_{k-i}| \leqslant (\sum_{k=n}^{\infty} d_k^2)^{1/2} (\sum_{k=n}^{\infty} d_{k-i}^2)^{1/2}$, Eq. (6) gives

$$\sum_{k=n}^{\infty} c_k^2 = O\left(\frac{1}{\log^{2+2\varepsilon} n}\right).$$

As a consequence (see [8, Theorem 3.3, p. 137]), $\sum_{n=0}^{\infty} c_n^2 \log^2 n < \infty$ and thus (see [8, Theorem 2.5, p. 126]), $\sum_{n=0}^{\infty} c_n \hat{B}_n(x)$ converges a.e. $x \in [-1, 1]$ to some function $g(x)$ (taking into account that $\|f\|_\mu^2 \leqslant \|f\|_{\mathbb{S}}^2$ for any $f \in \mathbb{S}$). But $f(x)$ belongs to $\mathbb{S}$ by continuity, so convergence in $\mathbb{S}$ of $\sum_{k=0}^{n} c_k \hat{B}_k(x)$ to $f(x)$ gives $g(x) = f(x)$ a.e. ∎

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